Keywords: Non-additive, Imprecise probability, Non-linear, Interval valued probability, Lower bound, Upper bound, Indeterminate.

SECTION 1. INTRODUCTION

Section Two will cover Keynes’s Part II analysis of his method of inexact measurement and approximation in the A Treatise on Probability (TP) and General Theory (GT), using interval valued probability and outcomes, based on Boole’s upper and lower bounded probabilities, which were non-additive. Section Three will cover the Heterodox, Institutionalist and Post Keynesian claims that Keynes’s probabilities are ordinals sometimes, but not if there are issues involving non-comparable, non measureable or incommensurable probabilities. This position overlooks the use of interval valued probability and decision weight analysis, as covered by Keynes in chapter 26 with his conventional coefficient of weight and risk, c, which accomplishes the same objective as interval valued probability, which is integrating non additivity into decision making. Section Four will cover the recent ‘model uncertainty’ literature of Hansen and Sargent (2019), which has unfortunately overlooked Keynes’s contributions in the TP that showed the deficiencies in Classical, Neoclassical, and New Neoclassical models, which, starting with Bentham, are all additive and linear in nature. Section Five will conclude the paper.
SECTION 2.KEYNES’S NON ADDITIVE APPROACH TO UNCERTAINTY USING INTERVAL VALUED PROBABILITY

It is in chapter XV of the TP, which is titled “Numerical Measurement and Approximation of Probabilities”, that Keynes presented his non additive approach to probability using imprecise probability in a systematic way:

“It is evident that the cases in which exact numerical measurement is possible are a very limited class, generally dependent on evidence which warrants a judgment of equi-probability by an application of the Principle of Indifference…. The sphere of inexact numerical comparison is not, however, quite so limited. Many probabilities, which are incapable of numerical measurement, can be placed nevertheless between (author’s note-Keynes’s emphasis) numerical limits. And by taking particular non-numerical probabilities as standards a great number of comparisons or approximate measurements become possible. If we can place a probability in an order of magnitude with some standard probability, we can obtain its approximate measure by comparison.

This method is frequently adopted in common discourse.”(Keynes, 1921, pp.159-160; boldface added).

It is Keynes’s worked out problem (Keynes, 1921, pp.162-163) of the TP, as well as the footnote on page 161 of the TP that discusses Boole’s technique in The Laws of Thought in chapters 16-21, that provides the detailed analysis on which the brief analysis, provided on pp.38-40 of chapter III (Keynes, 1921, pp.38-40), was presented as an introduction to Part II of the TP. The English mathematician, Wilbraham, showed decisively how Boole’s extremely difficult solutions approach could be made substantially clearer in an article published in 1854 in the Philosophical Magazine. However, Keynes had come up with his own approach to solving such problems :

“It is not worthwhile to work out more of these results here. Some less systematic approximations of the same kind are given in the course of the solutions in Chapter XVII. In seeking to compare the degree of one probability with that of another we may desire to get rid of one of the terms, on account of it’s not being comparable with any of our standard probabilities. Thus, our object in general is to eliminate a given symbol of quantity from a set of equations or inequalities. If, for instance, we are to obtain numerical limits within which our probability must lie, we must eliminate from the result those probabilities which are non-numerical.

This is the general problem for solution.(55) A general method of solving these problems when we can throw our equations into a linear shape so far as all symbols of probability are concerned, is best shown in the following example:...... where λ,μ,ν,ρ,σ,τ,υ represent probabilities which are to be eliminated, and limits are to be found for c in terms of the standard probabilities a, b, d, e, and l.”(Keynes,1921,pp.162-163)

Thus, the non-numerical probabilities λ,μ,ν,ρ,σ,τ,υ on (Keynes,1921,p.163) are just like the non-numerical probabilities V,Z, W,X,Y ,U specified in the diagram in (Keynes,1921, p.39) chapter III of the TP.

Keynes restricts this example to the case where “when we can throw our equations into a linear shape.”

However, Keynes makes it crystal clear that Boole’s technique works with nonlinear equations, as well in his footnote on page 161:

“*In Boole’s Calculus we are apt to be left with an equation of the second or of an even higher degree from which to derive the probability of the conclusion; and Boole introduced these methods in order to determine which of the several roots of his equation should be taken as giving the true solution of the problem in probability. In each case he shows that that root must be chosen which lies between certain limits, and that only one root satisfies this condition. The general theory to be applied in such cases is expounded by him in Chapter XIX. of The Laws of Thought, which is entitled “On Statistical Conditions.” But the solution given in that chapter is awkward and unsatisfactory, and he subsequently published a much better method in the Philosophical Magazine for 1854 (4th series, vol. viii.) under the title “On the Conditions by which the Solutions of Questions in the Theory of Probabilities are limited.”(Keynes, 1921, p.161).

Keynes deals with some of these problems in chapter 17.

All of the four curvatures in the diagram on p.39 of the TP are quadratic ,second order equations ,which Boole’s technique can be applied to in order to arrive at a root which has an upper and lower bound or limit.

The Keynesian fundamentalists (O’Donnell, Carabelli, Skidelsky, Meeks, etc.) claim , that the diagram on p.39 in chapter III of the TP is an illustration of an application of ordinal probability only, has no support at all once it is realized that Keynes is providing the detailed analysis in Part II of the TP and not Part I . The diagram on p.39 in chapter III of the TP was an introductory illustration of Keynes’s interval valued probability showing nonlinearity and non additivity that Keynes then explained in much greater detail in chapter XV of the TP. Chapter XV of the TP must be read and understood as supplying the final analysis promised by Keynes(Keynes,1921, pp.37-38) in the TP at the end of chapter III. Nowhere in chapter 15,16 or 17 of the TP is there any mention made of ordinal probability whatsoever. The same conclusion holds with respect to Boole’s 1854 The Laws of Thought.

In conclusion, the diagram on p.39 of the TP in chapter III can only be fully and completely understood after chapter XV of the TP has been fully absorbed .Keynes’s great concern with the addition property (additivity) in Part II simply means that any interpretation, that Keynes’s system of probability is ordinal, doesn’t make any sense because ordinal probability can’t be multiplied or added. It is an oxymoron to state that Keynes is concerned with the property of non-additivity and also simultaneously argue that his theory is an ordinal one because non-additivity means that the probabilities do not sum to 1.Ordinal probability has nothing to do with non-additivity.

Keynes’s alternative formulation to interval valued probability from Part II of the TP is his conventional coefficient of weight and risk, c, from chapter 26 of the TP, which is an alternative to the much more difficult interval valued proba-
bility analysis that Keynes provided in chapters 15 and 17 of the TP. The initial probabilities, p and q, are additive. Keynes then transforms the additive probabilities, p and q, by multiplying by decision weights. The conventional coefficient, c, results from the multiplication of p by \([2w/(1+w)]\) so as to introduce non-additivity and multiplying by \([1/(1+q)]\) in order to introduce non-linearity. It is impossible for p and q to be ordinal probabilities, since Keynes explicitly stated that they were additive. Ordinal probability can’t be added.

Finally, Keynes recognized that the common man uses interval valued probability, not ordinal probability, because it is simply far too weak to allow any sort of analysis which is supposed to serve as a guide to life.

**SECTION 3. THE MISINTERPRETATION AND MISREPRESENTATION OF KEYNES’S THEORY OF PROBABILITY**

The following description of Keynes’s contribution to imprecise probability in the appendix to the Stanford University entry on imprecise probability is presented below:

“The Keynes’ Theory of Probability contains the diagram reproduced in Figure H1, and it’s clear from this that he thought there could be degrees of belief that were not numerically comparable. Keynes interprets the O and I as the contradiction and tautology respectively and A is a proposition with a numerically measurable probability. The lines connect those propositions (denoted by letters) that can be compared. So V and W can be compared and W is more likely that V (since it is closer to I). Those propositions without lines between them (for example X and Y) are incomparable. Keynes’ own discussion of the diagram is on page 42 of Keynes (1921)." (Bradley, 2019).

The diagram on p.38 of the TP (p.43 of the 1973 CWJMK version of the TP) is presented and labeled as “…Figure H1: Keyes' view of probability.” This is complete nonsense and has nothing to do with Keynes’s view of probability. This complete misinterpretation and misrepresentation of Keynes’s view is actually based on the Keynesian Fundamentalists (Skidelsky, Carabelli, O’Donell—see references)view which skipped any reading of Part II of the TP.

Continuing, we find additional errors:

“Weatherston’s (2002) interprets Keynes as favouring some sort of IP view since sets of functions (or intervals of values) naturally give rise to the sorts of incomparabilities that Keynes takes to be features of belief. Keynes took (conditional) probability to be a sort of logical relationship that held between propositions (Hájek, 2011: 3.2), rather than as strength of belief. So whether Keynes would have approved of IP models is unclear. See Kyburg (2003) for a discussion of Keynes' view by someone sympathetic to IP."(Bradley, 2019).

Weatherston’s 2002 contribution suffers from the same severe and grave deficiencies as is present in the Keynesian fundamentalist literature, which has been accumulating since the late 1970’s, which is the failure to read Part II of the TP. No mention of Part II occurs in Weatherston’s article. There is no mention made of Keynes’s approach in Part II or III of the TP in any book or article published by Kyburg in his lifetime. Kyburg read only Part I of the TP. Contrary to Weatherston and Kyburg, Keynes presented an explicit, highly developed mathematical and logical theory of imprecise probability that has been completely overlooked except for an extremely small handful of academics (See Arthmar and Brady (2016,2017), Brady and Arthmar (2012), and Hailperin (1965,1986,1996) in the references).

Weatherston’s position is that some sense can be made out of Keynes’s strange and mysterious non numerical probabilities by reinterpreting them as interval probabilities based on the theory of imprecise probability. S. Brady’s acceptance of this statement at face value simply means that he, likewise, never covered Part II of the TP. Another recent academic article reaching the same erroneous conclusions as Weatherston (2002) and Bradley (2019) is Culham (2019, in press). There are a number of errors in his article:

“In the Treatise on Probability, Keynes (1921) introduces a truth relationship between proposition a and hypothesis h with probability p as a/h = p. For Keynes, however, new evidence, h1, even if it leaves the probability p unchanged, ‘increases the weight of the argument’ (Keynes, 1921, p. 71, original emphasis), meaning that the proposition is not more or less likely, but that the probability estimate is more reliable. ‘New evidence will sometimes decrease the probability of an argument, but it will always increase its “weight”’ (Keynes, 1921, p. 71). The new evidence must be ‘relevant’ in the sense that any part of the evidence can affect the probability, even if the new evidence as a whole and on balance does not (Keynes, 1921, pp. 55, 71–72). With this new evidence, the weight of the argument V (a/h) is increased to V (a/hh1). Keynes’s intuitive concept of weight can be reinterpreted to be reflected in the theory of imprecise probabilities (Weatherston, 2002), whereby weight can be represented by an interval of probabilities—the wider the interval, the more uncertain the event is, and the less is its weight.” (Culham, 2019, in press, section 5).

First, there is no truth relationship involved in Keynes’s logical probability relation \(P \), where \(P(a/h)=a,0\leq a\leq 1\), given that \(\alpha\) is a rational degree of probability which has nothing to do with the truth. Second, Culham is ignorant of the fact that V (a/h) = w, where \(0\leq w\leq 1\), where w represents the degree of the completeness of the evidence as discussed by Keynes on pp.314–315, which he has overlooked in his article. Third, the claim that “…Keynes’s intuitive concept of weight can be reinterpreted to be reflected in the theory of imprecise probabilities (Weatherston, 2002), whereby weight can be represented by an interval of probabilities—the wider the interval, the more uncertain the event is, and the less is its weight.” Was already made by both Boole and Keynes long ago. Keynes and Boole are the founders and creators of the theory of imprecise probabilities using interval valued probability with upper and lower bounded probabilities. Culham (2019), like Weatherston (2002), Bradley (2019), and many others both before and after them, have overlooked Part II of the TP:

“In the Treatise on Probability, Keynes (1921) introduces a truth relationship between proposition a and hypothesis h with probability p as a/h = p. For Keynes, however, new evi-
dence, h1, even if it leaves the probability p unchanged, ‘increases the weight of the argument’ (Keynes, 1921, p. 71, original emphasis), meaning that the proposition is not more or less likely, but that the probability estimate is more reliable. ‘New evidence will sometimes decrease the probability of an argument, but it will always increase its “weight”’ (Keynes, 1921, p. 71). The new evidence must be ‘relevant’ in the sense that any part of the evidence can affect the probability, even if the new evidence as a whole and on balance does not (Keynes, 1921, pp. 55, 71–72). With this new evidence, the weight of the argument V (a/h) is increased to V (a/h1). Keynes’s intuitive concept of weight can be reinterpreted to be reflected in the theory of imprecise probabilities (Weatherson, 2002). Keynes had already shown in 1921 how weight could be represented by interval probabilities, where the wider the interval, the more uncertain the proposition about the event is. Therefore, it has less weight.

SECTION 4. KEYNES, THE TP, AND MODEL UNCERTAINTY

Consider the following statement by (Hansen and Sargent, 2019)

“In a recent paper (Hansen and Sargent, 2019), we propose ways to categorize and respond to the multiple forms of uncertainty that confront decision makers and model builders. Thus, we distinguish among (1) uncertainty within a model; (2) uncertainty across a set of available known models; and (3) uncertainty about each model. We refer to (1) as risk – uncertainty about future outcomes that is described by a single known probability distribution. (This is the type of uncertainty assumed up until now in most work in theoretical and applied finance and macroeconomics.) We call uncertainty of type (2) ambiguity and represent it as being unsure about what weights or probabilities to attach to the available models. We call (3) model misspecification and represent it by surrounding each available model with a vast cloud statistical models with unknown forms that nevertheless fit the available data nearly as well as does an available model.” (Hansen and Sargent, 2019, p.2).

Based on this 3 way categorization, they arrive at the following literary descriptions of their Figures 1 and 2, which appear on pp. 3 and 4, respectively, of their paper. Consider their description of Figure 1:

“Figures 1 and 2 illustrate how we can depict key objects at play when investors make portfolio decisions while juggling our three types of model uncertainty about the multiple models that express their ‘views’ about the economy. Suppose that investors start with a simple linear first-order autoregressive model of macroeconomic growth rate dynamics but that their concerns about uncertainty induce them to explore other statistically similar specifications including ones that they especially fear. In Figure 1, the linear relation with the negative slope captures their baseline model with so-called mean reversion in growth rates. The mean reversion is evident because there is a pull from the more extreme growth states towards the centre of the growth rate distribution. The vertical axis is the local pull towards the centre of the distribution of macroeconomic growth (net of its long-run average growth rate.) If zero is the centre point, then in the absence of random shocks, there is a pull towards zero. The kinked line in red emerges when our investors consider formally many other possible and statistically similar specifications and when they then compute another model that investors fear might apply and that is statistically similar to the baseline linear model. The flatter slope to the left of zero reflects investors’ concerns in bad economic times that the macro economy may be stuck with more growth sluggishness than in the original model. The steeper slope to the right of zero reflects opposite forces. Here, good macroeconomic growth outcomes are feared to be shorter lived than in the original model specification. The blue and green curves show further adjustments induced by overall concerns that all of the models listed so far are wrong (misspecified). The overall downward shift occurs because the investors are averse to all of these different forms of uncertainty. “(Hansen and Sargent, 2019, p.3)

A very similar type of conclusion results from the application of Keynes’s conventional coefficient of weight and risk, c, as discussed by Keynes in chapter 26 of the TP on p.315 and in fl.2 on p. 315.

Keynes places p, probability, on the abscissa and c, the conventional coefficient decision weight, on the ordinate. If c=p , then you have the standard linear, additive model representation that describes the EMV and SEU decision theory approaches universally used in the economics profession. Now incorporate one after the other the Keynes’s weights [2w/(1+w)] and [p/(1+q)] that modify the linear and additive standard models to account for non additivity (ambiguity, vagueness, lack of confidence, etc.) and non linearity (nonlinear probability preferences), respectively. The various sets of nonlinear paths, for differing values of w and q used in the weights, curve away from the solid, linear, 45 degree line, where p=c holds. Keynes’s model was the first decision weighting model in history. Keynes created it to modify the standard linear and additive models being used by decision theorists in the first quarter of the twentieth century.

Now consider the literary description of their Figure 2:

“Figure 2 looks at how the implications are compounded over time by plotting deciles of implied macro consumption distributions over alternative horizons. For sake of simplicity, we depict only the results comparing the base line black line to the distributions associated with the green curve. The grey region depicts the baseline distribution, and the red region shows the impact of the uncertainty adjustment obtained by twisting probabilities in a conservative direction indicative of more cautious decision making in the face of uncertainty. This exercise holds fixed the constellation of their ‘views’. What varies, and what is being depicted in the figure, is the tractable way in which investors respond cautiously to the three types of uncertainty associated with their views.” (Hansen and Sargent, 2019, p.4).

Their Figure 2 leads to practically the same conclusions as would an application of Keynes’s interval valued analysis contained in Part II of the TP and discussed in chapter 4 of the GT on pp.39–40 and 43, which involve a reemphasis of Keynes’s method of inexact measurement and approximation presented in Part II of the TP. At the vertex of their figure, (0,0), one starts with a precise probability point value
such as \( p=0.5 \) being the probability of 3% GDP economic growth rate in the first quarter. As the number of GDP quarters starts to increase from the present to the immediate and/or near future, an imprecise, interval valued probability estimate starts to emerge, such as \( p \) being in the interval \((0.45, 0.55)\). This interval starts to increase as one moves toward an intermediate future with an interval such as \((0.35, 0.65)\). As one moves toward the far and distant future, the interval widens to \((0.25, 0.75)\). Once one is considering the far and distant future, the gap between the lower and upper bounds increases further, so that the estimates of a 3% growth rate becomes the interval \((0.15, 0.85)\). Of course, this interval represents uncertainty in the far and distant future 15-30 years out. Keynes argued that this was extremely important only in the specific case of long lived, physical, durable capital goods (factories and equipment), subject to the problem of technical obsolescence that resulted from future technological innovation, advance and change. The inherent difficulty of long run physical investment promoted, instead, the substitution of constant, short run speculation and financial manipulation in stock and money markets as an alternative option that would be substituted for investment in long run, extensive research and development. Confidence would thus decrease the farther out into the future you moved as uncertainty increased, leading to an upsurge in liquidity preference for the speculative demand for money, as opposed to the transactions demand for money. One moves toward the flat range of the LM curve (Keynes’s LP curve (Keynes, 1936, p. 199); see also Keynes’s discussions on (Keynes, 1936, pp 207-208) of the GT and his exchanges with J. Viner over the horizontal range of the LM(LP) curve in his 1937 reply article (Keynes, 1937, pp. 218-219).

Keynes specifically wrote Section 8 of chapter 12 of the GT to counter any claims about his ideas on uncertainty as being radical so that no long run macro equilibrium could exist. All formal and mathematical modeling methods became useless. Of course, this is the claim first made by Joan Robinson and G.L.S. Shackle, which was that no macroeconomic equilibrium can exist if there is uncertainty about the future.

An example of this approach leads to the following much distorted view of Keynes’s approach to modeling uncertainty and expectations with his inexact approach to measurement and approximation as discussed by Keynes in chapter four of the GT:

Consider the following statement made by L. B. Smaghi:

“In his Treatise on Probability... Keynes discussed three types of uncertainty. The first, and easiest to characterize and handle within macroeconomic models, is cardinal uncertainty. Its key feature is that it can be measured, and that a specific number can be attached to it. For example, we can say that, based on data from the last (say) thirty years, there is an X% probability that next year the euro area economy will grow by Y%. An important assumption is that the structure of the economy must remain sufficiently stable over time, or, in case it experiences time-variation, both the type of variation, and its pace (or velocity) must be known with a reasonable degree of accuracy.

The second type is ordinal uncertainty. In such a situation, although no specific number can be attached to the probability of a specific event, we are still in a position to state that event A is more likely than event B. According to Keynes this was by far the single most important probability class. It’s important to stress that this second type of uncertainty lies outside the dominant macroeconomic framework. Indeed, according to the rational expectations hypothesis, agents use the structure of the model in order to compute objective, numerical probabilities of alternative events occurring. Even in models where expectations are not fully rational, but rather computed on the basis of learning schemes, probabilities are computed econometrically.

Finally, there is the so-called ‘irreducible uncertainty’ – a concept originally introduced by Frank Knight... – which is simply based on the fact that agents have no rational basis for making any probabilistic statement of any kind about a specific event occurring or not occurring.

It is obviously very difficult to identify which type of uncertainty prevails at a specific point in time, especially with respect to this third one.

The economic literature confirms the uncertainty prevailing during those days. In their classic Monetary History of the United States, [8] Friedman and Schwartz pointed out that: “The contraction after 1929 clearly shattered beliefs in a “new era”, in the likelihood of long-continued stability [...]. The contraction instilled instead an exaggerated fear of continued economic instability, of the danger of stagnation, of the possibility of recurrent unemployment”. Events such as the 1929 stock market crash, or the September 2008 collapse of Lehman Brothers, create a vast amount of ‘irreducible uncertainty’, which exerts paralyzing effects on spending decisions and may cause the economy to slide into a depression.” (Smaghi, 2010, p.2).

Nowhere in Keynes’s A Treatise on Probability (Keynes, 1921) did Keynes deal with the types of categories specified by Smaghi (2010). The three categories Keynes dealt with are numerical probabilities, interval probabilities, and ignorance. Nowhere in Keynes’s work is there any reference to irreducible uncertainty unless one is dealing with the remote far and distant future. Keynes explicitly deals with ignorance (Keynes, 1921, pp. 309-312). Ignorance occurs if there is no relevant evidence, so that \( w=0 \). No conditional probabilities can be defined for \( P(a/h) = a \), because his non-existent. This is easily demonstrated either by Keynes’s interval valued probability approach as done above or by Hansen and Sargent (2019).

SECTION 5. CONCLUSIONS

It is in chapter 15 of the TP, (Keynes, 1921, pp. 161-163) as well as in chapters 16, 17, 20 and 22 of the TP, that Keynes provided a very detailed, mathematical analysis of his earlier, brief, graphical introductory exposition done in chapter III of the TP. Keynes, through his adaptation of Boole’s original approach, that Keynes had adapted and improved upon in the same manner that Wilbraham had first improved on the Boolean approach in 1854, allowed him to provide an explicit, imprecise approach to probability showing the importance of non-additivity in 1921.
In 1986, Theodore Hailperin showed decisively that the Boole-Keynes approach involved the use of an early version of linear programming techniques, using an initial basic, feasible solution, involving primal and dual maximization and minimization problems with constraints that were both equalities (linear) and inequalities (nonlinear) [Keynes’ inequations].

Keynes had already solved the mystery of the diagram on page 39 of chapter III in 1921. (Actually, the same analysis is provided by Keynes in the 1908 Cambridge Fellowship Dissertation. This Fellowship Dissertation would have earned Keynes a Ph.D. in Applied Mathematics at any University in the world at that time if he had been interested in obtaining a Ph.D., which he was not) in chapter 15 with additional problems solved in chapters 16, 17, 20, and 22 of the TP.

Keynes’s analysis (Keynes, 1921, pp.161-163) is straightforward if the reader of the TP has the requisite mathematical training and knowledge. It is highly likely that there is no economist, who has written on Keynes’s TP, who has such knowledge. This can be attributed to a reliance on Ramsey’s deficient reviews (See Ramsey, 1922, 1926)

The presentation of Keynes’s theory in the appendix to the entry on ‘Imprecise Probability’ in 2019 in the Stanford Encyclopedia of Philosophy is completely misleading. The failure to mention Boole’s path breaking work is a lacuna that can only be remedied by an extensive revision of the article. Hopefully, such a revision will finally set right the historical record regarding the original and detailed mathematical and logical contributions made by Keynes in 1907, 1908 and 1921, and by Boole in 1854.

CONFLICT OF INTEREST

There are no conflicts of interest

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Keynes’s Major Result from Part II of the A Treatise on Probability


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